PERGAMON

International Journal of Solids and Structures 38 (2001) 8773–8785

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Constitutive equations for transversely isotropic elastic dielectrics with Schur's lemma

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Received 18 August 2000; in revised form 3 April 2001

Abstract

Group-theoretic methods are used to impose symmetry restrictions on the form of the constitutive equations of linear theory of elastic dielectrics. A particular case of transversely isotropic elastic dielectrics which belong to the $D_6(622)$ crystal class and are governed by a group of 12 symmetry transformations and 6 irreducible representations associated with each element is discussed in detail. The product table is constructed and used to derive the basic quantities associated with the irreducible representations. Schur's lemma is applied to the constitutive equations and non-vanishing constants are determined from the linear algebraic systems and presented in tables. The number of independent material constants is reduced from 171 to 25. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Symmetry; Invariance; Schur's lemma; Elastic dielectrics

1. Introduction

Construction of constitutive equations of linear and non-linear electromagnetic–elastic crystals which remain invariant under a group of symmetry transformations has been an active area of research over the past few years. Finite group representations and various other procedures for determining the forms of polynomial constitutive expressions which describe the response of anisotropic materials to external forces have been developed by Voigt (1910), Nye (1986), Birss (1964), Smith and Rivlin (1958), Fumi (1952), Feischi and Fumi (1953). The use of continuous groups to construct integrity bases for isotropic materials is extensively discussed in Spencer (1971). Smith and Kiral (1978) based their approach on invariance and Schur's lemma to reduce coefficients in constitutive equations. The decomposition of magnetic material tensors into basic symmetry types which form the carrier spaces of irreducible representations associated with the group defining material symmetry is treated in Mert and Kiral (1977) and Kiral and Eringen (1990). For the two phases of KDP for which the material symmetry changes as the temperature increases through the 'Curie point', the constitutive equations are derived in Chowdhury and Glockner

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0020-7683/01/$ - see front matter

PII: S0020-7683(01)00104-4

(1982). Xu et al. (1987) have developed a computer aided procedure for the generation of constitutive expressions invariant under symmetry groups and cover the 32 conventional crystal classes.

In this paper, the methods of group representations theory and Schur's lemma are applied to derive the constitutive equations of transversely isotropic elastic dielectrics which belong to the $D_6(622)$ crystal class. An indexing scheme is used and the basic constitutive equations for the general linear elastic dielectrics are written in matrix form which contain six components each of stress and strain tensors (σ_{ij}, S_{ij}), three components of each of the electric and polarization vectors $({}_{\mathrm{L}}E_i, P_i)$, and nine components of each of the electric and polarization gradient tensors $(\varepsilon_{ij}, \Pi_{ij})$ involving 171 independent constants. The symmetry group for the $D_6(622)$ crystal class of transversely isotropic elastic dielectrics consists of twelve 3×3 orthogonal matrices and six irreducible representations associated with each of the elements. The product table is constructed and used in the decomposition of the strain tensor, the polarization vector and the polarization gradient tensor into linear combinations of their components which form the carrier spaces of irreducible representations. The invariance of the constitutive equations under the group of symmetry transformations and application of Schur's lemma lead to systems of linear algebraic equations in material constants which are solved and non-vanishing constants are determined. The number of independent constants is reduced from 171 to 25.

2. Basic equations

For a homogeneous elastic dielectric crystal which is bounded by a surface S and occupies a region V of a rectangular coordinate system, the general system of linear constitutive equations for the components of stress tensor σ_{ij}, the local electric vector ${}_{\mathrm{L}}E_i$ and electric tensor ε_{ij} are given by Mindlin (1968)

$$\sigma_{ij} = c_{ijkl}S_{kl} + f_{kij}P_k + d_{klij}\Pi_{kl} \tag{2.1}$$

$$-{}_{\mathrm{L}}E_i = f_{ikl}S_{kl} + a_{ik}P_k + j_{ikl}\Pi_{kl} \tag{2.2}$$

$$\varepsilon_{ij} = d_{ijkl}S_{kl} + j_{kij}P_k + b_{ijkl}\Pi_{kl} \tag{2.3}$$

where S_{ij} are the components of the symmetric strain tensor, P_i are the components of polarization vector, and $\Pi_{ij} = P_{i,j}$ are the components of polarization gradient tensor. The coefficient tensors c_{ijkl}, a_{ij}, b_{ijkl}, f_{ijk}, j_{ijk} and d_{ijkl} are the constant elastic and dielectric tensors.

We use the abbreviated notation and express the components of tensors σ_{ij}, S_{ij}, Π_{ij} and ε_{ij} as vectors which are written as

$$\boldsymbol{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}]^{\mathrm{t}}, \quad \mathbf{S} = [S_{11}, S_{22}, S_{33}, S_{23}, S_{31}, S_{12}]^{\mathrm{t}}$$

$$\boldsymbol{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}, \varepsilon_{32}, \varepsilon_{31}, \varepsilon_{13}, \varepsilon_{12}, \varepsilon_{21}]^{\mathrm{t}}$$

$$\boldsymbol{\Pi} = [\Pi_{11}, \Pi_{22}, \Pi_{33}, \Pi_{23}, \Pi_{32}, \Pi_{31}, \Pi_{13}, \Pi_{12}, \Pi_{21}]^{\mathrm{t}} \tag{2.4}$$

where the superscript 't' denotes the transpose of the row vector. The scheme for indexing the coefficient tensors in which a pair of indices ij or kl is replaced by a single index is indicated in Appendix A.

The system of constitutive equations, Eqs. (2.1)–(2.3) can now be written in the matrix form as

$$\begin{bmatrix} \underset{(6\times1)}{\boldsymbol{\sigma}} \\ -{}_{\mathrm{L}}\underset{(3\times1)}{\mathbf{E}} \\ \underset{(9\times1)}{\boldsymbol{\varepsilon}} \end{bmatrix} = \begin{bmatrix} \underset{(6\times6)}{\mathbf{c}} & \underset{(6\times3)}{\mathbf{f}^{\mathrm{t}}} & \underset{(6\times9)}{\mathbf{d}^{\mathrm{t}}} \\ \underset{(3\times6)}{\mathbf{f}} & \underset{(3\times3)}{\mathbf{a}} & \underset{(3\times9)}{\mathbf{j}} \\ \underset{(9\times6)}{\mathbf{d}} & \underset{(9\times3)}{\mathbf{j}^{\mathrm{t}}} & \underset{(9\times9)}{\mathbf{b}} \end{bmatrix} \begin{bmatrix} \underset{(6\times1)}{\mathbf{S}} \\ \underset{(3\times1)}{\mathbf{P}} \\ \underset{(9\times1)}{\boldsymbol{\Pi}} \end{bmatrix} \tag{2.5}$$

where the numbers in parentheses below the matrix indicate the order of the matrix. For an elastic dielectric, the total number of independent material constants given by matrices **c**, **a**, **b**, **j**, **d**, **f** equals 171.

Let the symmetry properties of a given crystal class be defined by the set of 3×3 orthogonal matrices

$$\mathbf{A}_1 = \mathbf{I} = [\delta_{ij}], \quad \mathbf{A}_2 = [A_{ij}^{(2)}], \ldots, \mathbf{A}_N = [\mathbf{A}_{ij}^{(N)}] \tag{2.6}$$

The symmetry transformations $\mathbf{A}_K$ $(K = 1, 2, \ldots, N)$ carry the components of the unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of a rectangular coordinate system into unit vectors $\{\mathbf{e}_1^{(K)}, \mathbf{e}_2^{(K)}, \mathbf{e}_3^{(K)}\}$ and the components of the tensors S_{ij}, P_i, Π_{ij} are transformed as

$$e_i^{(K)} = A_{ij}^{(K)} e_j, \quad P_i^{(K)} = A_{ij}^{(K)} P_j \tag{2.7}$$

$$S_{ij}^{(K)} = \mathbf{A}_{im}^{(K)} A_{jn}^{(K)} S_{mn}, \quad \Pi_{ij}^{(K)} = A_{im}^{(K)} A_{jn}^{(K)} \Pi_{mn} \quad (K = 1, 2, \ldots, N) \tag{2.8}$$

By listing the components of the tensors in the frames of reference $\mathbf{e}^{(K)}$ and $\mathbf{e}$ according to the notation in Eq. (2.4) and using Eqs. (2.7) and (2.8), we can construct matrices $\mathbf{T}^{S(K)}$, $\mathbf{T}^{P(K)}$, $\mathbf{T}^{\Pi(K)}$ $(K = 1, 2, \ldots, N)$ of orders (6×6), (3×3) and (9×9) respectively, such that

$$(\boldsymbol{\sigma}^{(K)}, \mathbf{S}^{(K)}) = \mathbf{T}^{S(K)}(\boldsymbol{\sigma}, \mathbf{S}), \quad ({}_{\mathrm{L}}\mathbf{E}^{(K)}, \mathbf{P}^{(K)}) = \mathbf{T}^{P(K)}({}_{\mathrm{L}}\mathbf{E}, \mathbf{P})$$

$$(\boldsymbol{\varepsilon}^{(K)}, \boldsymbol{\Pi}^{(K)}) = \mathbf{T}^{\boldsymbol{\Pi}^{(K)}}(\boldsymbol{\varepsilon}, \boldsymbol{\Pi}), \quad (K = 1, 2, \ldots, N) \tag{2.9}$$

The constitutive equations Eq. (2.5) remain invariant under the group of symmetry transformations $\{\mathbf{A}\} = \{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N\}$ if

$$\begin{bmatrix} \boldsymbol{\sigma}^{(K)} \\ -{}_{\mathrm{L}}\mathbf{E}^{(K)} \\ \boldsymbol{\varepsilon}^{(K)} \end{bmatrix} = \begin{bmatrix} \mathbf{c} & \mathbf{f}^{\mathrm{t}} & \mathbf{d}^{\mathrm{t}} \\ \mathbf{f} & \mathbf{a} & \mathbf{j} \\ \mathbf{d} & \mathbf{j}^{\mathrm{t}} & \mathbf{b} \end{bmatrix} \begin{bmatrix} \mathbf{S}^{(K)} \\ \mathbf{P}^{(K)} \\ \boldsymbol{\Pi}^{(K)} \end{bmatrix} \quad (K = 1, 2, \ldots, N) \tag{2.10}$$

From Eqs. (2.5), (2.9) and (2.10), the invariance of the constitutive equations leads to the following restrictions on the material coefficient matrices

$$\begin{bmatrix} \mathbf{T}^{S(K)} & \cdot & \cdot \\ \cdot & \mathbf{T}^{P(K)} & \cdot \\ \cdot & \cdot & \mathbf{T}^{\Pi(K)} \end{bmatrix} \begin{bmatrix} \mathbf{c} & \mathbf{f}^{\mathrm{t}} & \mathbf{d}^{\mathrm{t}} \\ \mathbf{f} & \mathbf{a} & \mathbf{j} \\ \mathbf{d} & \mathbf{j}^{\mathrm{t}} & \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{c} & \mathbf{f}^{\mathrm{t}} & \mathbf{d}^{\mathrm{t}} \\ \mathbf{f} & \mathbf{a} & \mathbf{j} \\ \mathbf{d} & \mathbf{j}^{\mathrm{t}} & \mathbf{b} \end{bmatrix} \begin{bmatrix} \mathbf{T}^{S(K)} & \cdot & \cdot \\ \cdot & \mathbf{T}^{P(K)}. & \\ \cdot & \cdot & \mathbf{T}^{\Pi(K)} \end{bmatrix} \quad (K = 1, 2, \ldots, N) \tag{2.11}$$

where '·' denotes the zero matrix.

Let $\{\Gamma_i^{(K)}\}(i = 1, 2, 3, \ldots, r)$ $(K = 1, 2, \ldots, N)$ be the r inequivalent irreducible representations associated with the $\mathbf{A}_K$ $(K = 1, 2, \ldots, N)$ of the crystallographic group $\{\mathbf{A}\}$. The basic quantities $U_1^S, U_2^S, U_3^S, \ldots$; $U_1^P, U_2^P, U_3^P, \ldots$; $U_1^\Pi, U_2^\Pi, U_3^\Pi, \ldots$; are the linear combination of the components of the vectors **S**, **P**, **Π** and are said to be the carrier spaces of the irreducible representations of the group. These are determined from the formula Lomont (1959).

$$U_p^R = \sum_{K=1}^{N} \Gamma^K T_{pq}^{R(K)} R_q \quad (\mathbf{R} = \mathbf{S}, \mathbf{P}, \boldsymbol{\Pi}) \tag{2.12}$$

An alternative procedure for finding basic quantities is based on the product tables and is described in Xu et al. (1987). The matrices $\mathbf{Q}^S$, $\mathbf{Q}^P$, $\mathbf{Q}^\Pi$ can be easily constructed from the basic quantities such that the vectors

$$\mathbf{U}^R = \mathbf{Q}^R \mathbf{R} \quad (\mathbf{R} = \mathbf{S}, \mathbf{P}, \boldsymbol{\Pi}) \tag{2.13}$$

list the basic quantities in the order of the irreducible representations. It has been observed (Lomont, 1959) that the matrices $\mathbf{Q}^R$ reduce the matrices $\mathbf{Q}^R\mathbf{T}^{R(K)}\mathbf{Q}^{-1R}(\mathbf{R} = \mathbf{S}, \mathbf{P}, \mathbf{\Pi})$ to the direct sum of irreducible representations of $\mathbf{A}_K$, the number of its occurrence is given by the formula (Lomont, 1959)

$$\alpha_r^{\mathrm{T}^R}(\Gamma_j) = \frac{1}{N}\sum_{K=1}^{N}\mathrm{tr}(\mathbf{T}^{R(K)})\mathrm{tr}(\Gamma_j^{(K)}) \tag{2.14}$$

where the suffix tr before the matrix stands for its trace.

Schur's lemma states that

1. If Γ^K is an irreducible representation of dimension n of group G consisting of matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N$ and if an $n \times n$ matrix $\mathbf{M}$ commutes with each of the matrices Γ^{K}, i.e.

$$\Gamma^K\mathbf{M} = \mathbf{M}\Gamma^K \quad (K = 1, \ldots, N)$$

then $\mathbf{M} = \lambda\mathbf{I}$, where λ is a constant and $\mathbf{I}$ is the $n \times n$ identity matrix.

2. If Γ_1^K and Γ_2^K are two irreducible representations of dimensions n and m respectively of the group G and if there is a $n \times m$ matrix $\mathbf{M}$ such that

$$\Gamma_1^K\mathbf{M} = \mathbf{M}\Gamma_2^K \quad (K = 1, \ldots, N)$$

then $\mathbf{M}$ must be a zero matrix or $n = m$ with $\mathbf{M}$ a non-singular matrix.

The non-vanishing material constants in the constitutive equations are determined by using the invariance conditions Eq. (2.11), the matrices $\mathbf{Q}^R$ associated with the basic quantities and the Schur's lemma.

3. Basic quantities for the $D_6(622)$ class

Dielectrics subjected to electric voltage at high temperature are known to become transversely isotropic (Mason, 1966). In this section, we give in detail the procedure for the construction of constitutive equations of transversely isotropic elastic dielectrics which belong to the $D_6(622)$ crystal class.

The matrices comprising the symmetry group of this crystal class are given by Kiral and Smith (1974)

$$\begin{aligned}
&\mathbf{A}_1 = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & \cdot \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \cdot \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}, \\
&\mathbf{A}_4 = \begin{bmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}, \quad \mathbf{A}_5 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & \cdot \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}, \quad \mathbf{A}_6 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & \cdot \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}, \\
&\mathbf{A}_7 = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{bmatrix}, \quad \mathbf{A}_8 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & \cdot \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & \cdot \\ \cdot & \cdot & -1 \end{bmatrix}, \quad \mathbf{A}_9 = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \cdot \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \cdot \\ \cdot & \cdot & -1 \end{bmatrix}, \\
&\mathbf{A}_{10} = \begin{bmatrix} -1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -1 \end{bmatrix}, \quad \mathbf{A}_{11} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & \cdot \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \cdot \\ \cdot & \cdot & -1 \end{bmatrix}, \quad \mathbf{A}_{12} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & \cdot \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & \cdot \\ \cdot & \cdot & -1 \end{bmatrix}
\end{aligned} \tag{3.1}$$

There are six inequivalent irreducible representations (Kiral and Smith, 1974) $\Gamma_i^K (i = 1\text{–}6,\ K = 1\text{–}12)$ associated with the group and are of degree one and two as listed in Table 1.

Table 1
Irreducible representations of $D_6(622)$ class

	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$	$\mathbf{A}_4$	$\mathbf{A}_5$	$\mathbf{A}_6$	$\mathbf{A}_7$	$\mathbf{A}_8$	$\mathbf{A}_9$	$\mathbf{A}_{10}$	$\mathbf{A}_{11}$	$\mathbf{A}_{12}$
Γ_1	1	1	1	1	1	1	1	1	1	1	1	1
Γ_2	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1
Γ_3	1	1	1	−1	−1	−1	1	1	1	−1	−1	−1
Γ_4	1	1	1	−1	−1	−1	−1	−1	−1	1	1	1
Γ_5	**E**	**A**	**B**	−**E**	−**A**	−**B**	**F**	**G**	**H**	−**F**	−**G**	−**H**
Γ_6	**E**	**A**	**B**	**E**	**A**	**B**	−**F**	−**G**	−**H**	−**F**	−**G**	−**H**

E, **A**, **B**, **F**, **G**, **H** are 2×2 matrices.

In Table 1

$$\mathbf{E} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

The representations $\mathbf{T}^{R(K)}$ $(\mathbf{R} = \mathbf{S}, \mathbf{P}, \mathbf{\Pi})$ which give the transformation properties of vectors **S**, **P**, **Π** under the group $\{\mathbf{A}\}$ can be determined from Eqs. (2.7)–(2.9).

The Product Table: Let

$$\begin{aligned} &\Gamma_1 : a_{11}, b_{11} \\ &\Gamma_2 : a_{21}, b_{21} \\ &\Gamma_3 : a_{31}, b_{31} \\ &\Gamma_4 : a_{41}, b_{41} \\ &\Gamma_5 : \begin{bmatrix} a_{51} \\ a_{52} \end{bmatrix}, \quad \begin{bmatrix} b_{51} \\ b_{52} \end{bmatrix} \\ &\Gamma_6 : \begin{bmatrix} a_{61} \\ a_{62} \end{bmatrix}, \quad \begin{bmatrix} b_{61} \\ b_{62} \end{bmatrix} \end{aligned} \tag{3.2}$$

denote quantities whose transformation properties under $D_6(622)$ crystal class are defined by Γ_i^K $(i = 1\text{–}6, K = 1, \ldots, 12)$. We consider 64 products $a_{11}b_{11}, \ldots, a_{62}b_{62}$ of the quantities appearing in Eq. (3.2). The linear combinations of these products may be split into 44 sets as follows:

$$\begin{aligned} &\Gamma_1 : a_{11}b_{11},\ a_{21}b_{21},\ a_{31}b_{31},\ a_{41}b_{41},\ a_{51}b_{51} + a_{52}b_{52},\ a_{61}b_{61} + a_{62}b_{62} \\ &\Gamma_2 : a_{11}b_{21},\ a_{21}b_{11},\ a_{31}b_{41},\ a_{41}b_{31},\ a_{51}b_{52} - a_{52}b_{51},\ a_{61}b_{62} - a_{62}b_{61} \\ &\Gamma_3 : a_{11}b_{31},\ a_{31}b_{11},\ a_{21}b_{41},\ a_{41}b_{21},\ a_{51}b_{62} - a_{52}b_{61},\ a_{61}b_{52} - a_{62}b_{51} \\ &\Gamma_4 : a_{11}b_{41},\ a_{41}b_{11},\ a_{21}b_{31},\ a_{31}b_{21},\ a_{51}b_{61} + a_{52}b_{62},\ a_{61}b_{51} + a_{62}b_{52} \\ &\Gamma_5 : \begin{bmatrix} a_{11}b_{51} \\ a_{11}b_{52} \end{bmatrix}, \begin{bmatrix} a_{51}b_{11} \\ a_{52}b_{11} \end{bmatrix}, \begin{bmatrix} a_{21}b_{52} \\ -a_{21}b_{51} \end{bmatrix}, \begin{bmatrix} a_{52}b_{21} \\ -a_{51}b_{21} \end{bmatrix}, \begin{bmatrix} a_{31}b_{62} \\ -a_{31}b_{61} \end{bmatrix}, \begin{bmatrix} a_{62}b_{31} \\ -a_{61}b_{31} \end{bmatrix}, \\ &\quad \begin{bmatrix} a_{41}b_{62} \\ -a_{41}b_{61} \end{bmatrix}, \begin{bmatrix} a_{62}b_{41} \\ -a_{61}b_{41} \end{bmatrix}, \begin{bmatrix} a_{51}b_{62} + a_{52}b_{61} \\ a_{51}b_{61} - a_{52}b_{62} \end{bmatrix}, \begin{bmatrix} a_{61}b_{52} + a_{62}b_{51} \\ a_{61}b_{51} - a_{62}b_{52} \end{bmatrix} \\ &\Gamma_6 : \begin{bmatrix} a_{11}b_{61} \\ a_{11}b_{62} \end{bmatrix}, \begin{bmatrix} a_{61}b_{11} \\ a_{62}b_{11} \end{bmatrix}, \begin{bmatrix} a_{21}b_{62} \\ -a_{21}b_{61} \end{bmatrix}, \begin{bmatrix} a_{62}b_{21} \\ -a_{61}b_{21} \end{bmatrix}, \begin{bmatrix} a_{31}b_{52} \\ -a_{31}b_{51} \end{bmatrix}, \\ &\quad \begin{bmatrix} a_{52}b_{31} \\ -a_{51}b_{31} \end{bmatrix}, \begin{bmatrix} a_{41}b_{52} \\ -a_{41}b_{51} \end{bmatrix}, \begin{bmatrix} a_{52}b_{41} \\ -a_{51}b_{41} \end{bmatrix}, \begin{bmatrix} a_{51}b_{52} + a_{52}b_{51} \\ a_{51}b_{51} - a_{52}b_{52} \end{bmatrix}, \begin{bmatrix} a_{61}b_{62} + a_{62}b_{61} \\ a_{61}b_{61} - a_{62}b_{62} \end{bmatrix} \end{aligned} \tag{3.3}$$

Table 2
Carrier spaces of the irreducible representations of D_6(622) class

	S	**P**	**Π**
Γ_1	$S_{33};\ S_{11}+S_{22}$	0	$\Pi_{33};\ \Pi_{11}+\Pi_{22}$
Γ_2	0	P_3	$\Pi_{12}-\Pi_{21}$
Γ_5	$[S_{23},\ -S_{13}]^t$	$[P_1,\ P_2]^t$	$[\Pi_{23},-\Pi_{13}]^t,\ [\Pi_{32},\ -\Pi_{31}]^t$
Γ_6	$[2S_{12}, S_{11}-S_{22}]^t$	0	$[\Pi_{12}+\Pi_{21},\ \Pi_{11}-\Pi_{22}]^t$

The transformation properties of these products are defined by Γ_i ($i = 1$–6). It may be verified from expressions (3.2) and (3.3) that

$$\left[a_{11}^K b_{11}^K\right] = \Gamma_1^K [a_{11} b_{11}] \tag{3.4}$$

$$\begin{bmatrix} a_{51}^K b_{62}^K + a_{52}^K b_{61}^K \\ a_{51}^K b_{61}^K - a_{52}^K b_{62}^K \end{bmatrix} = \Gamma_5^K \begin{bmatrix} a_{51} b_{62} + a_{52} b_{61} \\ a_{51} b_{61} - a_{52} b_{62} \end{bmatrix} \tag{3.5}$$

The other product combinations can be verified in a similar manner.

We use the product table (3.3) to construct the basic quantities which are the carrier spaces of the irreducible representations. The components of an absolute vector $\mathbf{v} = [v_1, v_2, v_3]$ under the D_6(622) class transform according to the rule (Kiral and Smith, 1974)

$$\Gamma_2 : v_3; \quad \Gamma_5 : [v_1, v_2]^t$$

The decomposition of the nine components of the second order non-symmetric tensor $\Pi_{ij} = u_i v_j$ can be derived from the product table by assuming

$$u_3 = a_{21},\ \ v_3 = b_{21},\ \ [u_1, u_2] = [a_{51}, a_{52}],\ \ [v_1, v_2] = [b_{51}, b_{52}] \tag{3.6}$$

and all other a and b are assumed to vanish. The decomposition for the symmetric tensor S_{ij} can be derived by assuming $S_{ij} = v_i v_j$. The irreducible representations and the associated basic quantities are listed in Table 2.

4. Reduction of constitutive coefficients by Schur's lemma

From Eq. (2.13) and Table 2 for the basic quantities and the order of components in vectors $\mathbf{R}$ and $\mathbf{U}^R$, we construct the matrices $\mathbf{Q}^R$ ($\mathbf{R} = \mathbf{S}, \mathbf{P}, \mathbf{\Pi}$). These matrices and their inverses are listed in Appendix B. It is easily seen that

$$\mathbf{Q}^S \mathbf{T}^{S(K)} \mathbf{Q}^{S-1} = 2\Gamma_1^K \dot{+} \Gamma_5^K \dot{+} \Gamma_6^K \tag{4.1}$$

$$\mathbf{Q}^P \mathbf{T}^{P(K)} \mathbf{Q}^{P-1} = \Gamma_2^K \dot{+} \Gamma_5^K \tag{4.2}$$

$$\mathbf{Q}^\Pi \mathbf{T}^{\Pi(K)} \mathbf{Q}^{\Pi-1} = 2\Gamma_1^K \dot{+} \Gamma_2^K \dot{+} 2\Gamma_5^K \dot{+} \Gamma_6^K \quad (K = 1\text{–}12) \tag{4.3}$$

where the symbol $\dot{+}$ represents the direct sum of irreducible representations.

Pre-multiplying and post-multiplying Eq. (2.11) by

$$\begin{bmatrix} \mathbf{Q}^S & \cdot & \cdot \\ \cdot & \mathbf{Q}^P & \cdot \\ \cdot & \cdot & \mathbf{Q}^\Pi \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{Q}^{S-1} & \cdot & \cdot \\ \cdot & \mathbf{Q}^{P-1} & \cdot \\ \cdot & \cdot & \mathbf{Q}^{\Pi-1} \end{bmatrix}$$

and using Eqs. (4.1)–(4.3), the matrix of coefficients Eq. (2.11) satisfies

$$\Gamma^K \mathbf{D} = \mathbf{D} \Gamma^K \tag{4.4}$$

where

$$\Gamma^K = 2\Gamma_1^K \dot{+} \Gamma_5^K \dot{+} \Gamma_6^K \dot{+} \Gamma_2^K \dot{+} \Gamma_5^K \dot{+} 2\Gamma_1^K \dot{+} \Gamma_2^K \dot{+} 2\Gamma_5^K \dot{+} \Gamma_6^K \tag{4.5}$$

$$\mathbf{D} = \begin{bmatrix} \mathbf{Q}^S & \cdot & \cdot \\ \cdot & \mathbf{Q}^P & \cdot \\ \cdot & \cdot & \mathbf{Q}^\Pi \end{bmatrix} \begin{bmatrix} \mathbf{c} & \mathbf{f}^t & \mathbf{d}^t \\ \mathbf{f} & \mathbf{a} & \mathbf{j} \\ \mathbf{d} & \mathbf{j}^t & \mathbf{b} \end{bmatrix} \begin{bmatrix} \mathbf{Q}^{S-1} & \cdot & \cdot \\ \cdot & \mathbf{Q}^{P-1} & \cdot \\ \cdot & \cdot & \mathbf{Q}^{\Pi-1} \end{bmatrix} \tag{4.6}$$

Eq. (4.4) is of the form that admits application of Schur's lemma. The sub-matrices of **D** are partitioned consistent with the irreducible representations. A direct comparison of the corresponding terms of matrices is given in Appendix C. It is seen that some constants vanish and others satisfy linear equations. Twenty five non-vanishing independent elastic and dielectric constants found are given by

$$c_{11} = c_{22},\ \ c_{12}, c_{13} = c_{23},\ \ c_{33}, c_{44} = c_{55},\ \ c_{66} = (c_{11} - c_{12});$$

$$a_{33}, a_{11} = a_{22};\ \ f_{14} = -f_{25};\ \ j_{38} = -j_{39},\ \ j_{14} = -j_{27},\ \ j_{15} = -j_{26};$$

$$d_{33}, d_{12} = d_{21},\ \ d_{13} = d_{23},\ \ d_{31} = d_{32},\ \ d_{11} = d_{22},$$

$$d_{54} = d_{65},\ \ d_{44} = d_{75},\ \ d_{86} = d_{96} = d_{11} - d_{12};$$

$$b_{33}, b_{12} = b_{21},\ \ b_{11} = b_{22},\ \ b_{31} = b_{32},\ \ b_{55} = b_{66},\ \ b_{44} = b_{77},$$

$$b_{88} = b_{99} = b_{11} - b_{12};\ \ b_{54} = b_{67}$$

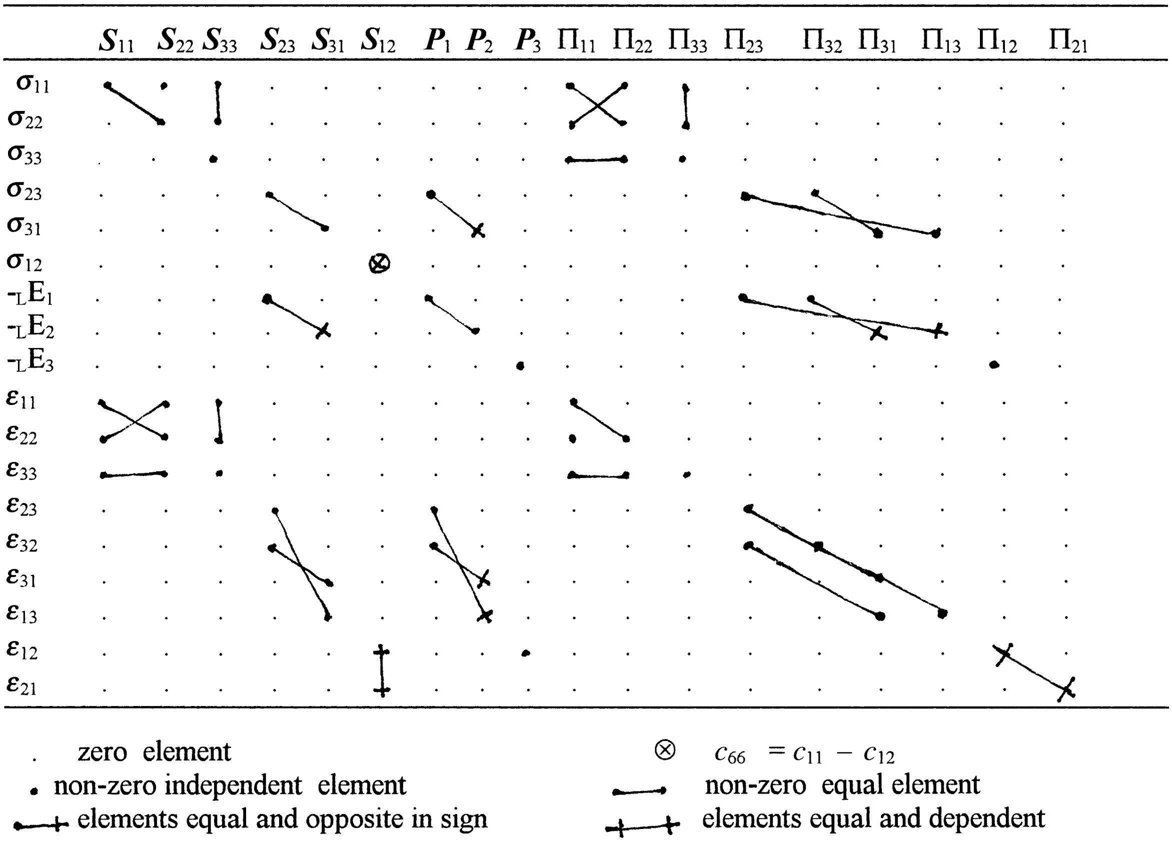
The number of independent elastic and dielectric constants with no symmetry and those belonging to $D_6(622)$ symmetry class are listed in Table 3.

The constitutive equations and the non-vanishing independent material constants are explicitly shown in Table 4.

Table 3
Number of independent constants in coefficient matrices

Matrix	No symmetry	D_6 symmetry
a	6	2
b	45	7
c	21	5
d	54	7
f	18	1
j	27	3
Total	171	25

Table 4
Constitutive equations for elastic dielectrics in $D_6(622)$ class



We observe that in the natural state of elastic dielectric when there is no symmetry the number of material constants equals 171. For the transversely isotropic elastic dielectrics of $D_6(622)$ class the number of material constants is reduced from 171 to 25.

Appendix A

Indexing scheme for c_{ijkl} and f_{mij} $(m = 1, 3)$:

(ij), (kl)	11	22	33	23,32	31,13	12,21
	1	2	3	4	5	6

Indexing scheme for b_{ijkl}, j_{mkl} $(m = 1, 6)$:

(ij), (kl)	11	22	33	23	32	31	13	12	21
	1	2	3	4	5	6	7	8	9

Indexing scheme for d_{klij}:

(kl)	11	22	33	23	32	31	13	12	21
	1	2	3	4	5	6	7	8	9
(ij)	11	22	33	23,32	31,13	12,21			
	1	2	3	4	5	6			

Appendix B

The matrices $\mathbf{Q}^S$, $\mathbf{Q}^P$, $\mathbf{Q}^\Pi$ which transform the vectors $\mathbf{S}$, $\mathbf{P}$, Π to their linear combinations and form the basic quantities for the irreducible representations and their inverses are listed as:

$$Q^S = \begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 \\ 1 & -1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad Q^{S-1} = \begin{bmatrix} \cdot & \frac{1}{2} & \cdot & \cdot & \cdot & \frac{1}{2} \\ \cdot & \frac{1}{2} & \cdot & \cdot & \cdot & -\frac{1}{2} \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \frac{1}{2} & \cdot \end{bmatrix},$$

$$Q^P = \begin{bmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{bmatrix}, \quad Q^{P-1} = \begin{bmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \end{bmatrix},$$

$$Q^\Pi = \begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ 1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad Q^{\Pi-1} = \begin{bmatrix} \cdot & \frac{1}{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{2} \\ \cdot & \frac{1}{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{1}{2} \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{2} & \cdot & \cdot & \cdot & \cdot & \frac{1}{2} & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot & \cdot & \cdot & \cdot & \frac{1}{2} & \cdot \end{bmatrix}$$

Appendix C

The six sub-matrices of matrix $\mathbf{D}$ in Eq. (4.4) are partitioned consistent with the irreducible representations in Γ and comparison of corresponding elements and application of Schur's lemma leads to constraints on the elastic and dielectric constants. The details are as follows:

(1) The matrix $\mathbf{Q}^S\mathbf{c}\mathbf{Q}^{S-1}$:

Let

$$\mathbf{c}^* = \mathbf{Q}^S\mathbf{c}\mathbf{Q}^{S-1} = \begin{bmatrix} c^*_{11} & c^*_{12} & c^*_{13} & c^*_{14} \\ c^*_{21} & c^*_{22} & c^*_{23} & c^*_{24} \\ c^*_{31} & c^*_{32} & c^*_{33} & c^*_{34} \\ c^*_{41} & c^*_{42} & c^*_{43} & c^*_{44} \end{bmatrix},$$

where $c^*_{11}, c^*_{12}, c^*_{21}, c^*_{22}$ are 1×1 matrices; $c^*_{13}, c^*_{14}, c^*_{23}, c^*_{24}$ are 1×2 matrices; $c^*_{31}, c^*_{32}, c^*_{41}, c^*_{42}$ are 2×1 matrices; c^*_{34}, c^*_{43}, c^*_{33}, c^*_{44} are 2×2 matrices which can be written in terms of elements of 6×6 matrix **c**.

From Eq. (4.4), we derive

$$\begin{bmatrix} \Gamma^K_1 c^*_{11} & \Gamma^K_1 c^*_{12} & \Gamma^K_1 c^*_{13} & \Gamma^K_1 c^*_{14} \\ \Gamma^K_1 c^*_{21} & \Gamma^K_1 c^*_{22} & \Gamma^K_1 c^*_{23} & \Gamma^K_1 c^*_{24} \\ \Gamma^K_5 c^*_{31} & \Gamma^K_5 c^*_{32} & \Gamma^K_5 c^*_{33} & \Gamma^K_5 c^*_{34} \\ \Gamma^K_6 c^*_{41} & \Gamma^K_6 c^*_{42} & \Gamma^K_6 c^*_{43} & \Gamma^K_6 c^*_{44} \end{bmatrix} = \begin{bmatrix} c^*_{11}\Gamma^K_1 & c^*_{12}\Gamma^K_1 & c^*_{13}\Gamma^K_5 & c^*_{14}\Gamma^K_6 \\ c^*_{21}\Gamma^K_1 & c^*_{22}\Gamma^K_1 & c^*_{23}\Gamma^K_5 & c^*_{24}\Gamma^K_6 \\ c^*_{31}\Gamma^K_1 & c^*_{32}\Gamma^K_1 & c^*_{33}\Gamma^K_5 & c^*_{34}\Gamma^K_6 \\ c^*_{41}\Gamma^K_1 & c^*_{42}\Gamma^K_1 & c^*_{43}\Gamma^K_5 & c^*_{44}\Gamma^K_6 \end{bmatrix} \tag{C.1}$$

Comparing the corresponding terms from Eq. (C.1) and applying Schur's lemma, we find the following results.

(i) $c^*_{11}, c^*_{12}, c^*_{21}, c^*_{22}$ are 1×1 non-zero matrices.
(ii) c^*_{33}, c^*_{44} are 2×2 matrices of the form $\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, λ is a constant.
(iii) All other matrices in c^* vanish.

The system of linear equations derived from Eq. (C.1) can be solved and leads to five independent constants:

$$c_{11} = c_{22}, \quad c_{12}, c_{13} = c_{23}, \quad c_{33}, c_{44} = c_{55}, \quad c_{66} = (c_{11} - c_{12})$$

(2) The matrix $\mathbf{Q}^P\mathbf{f}\mathbf{Q}^{S-1}$

Let

$$f^* = \mathbf{Q}^P\mathbf{f}\mathbf{Q}^{S-1} = \begin{bmatrix} f^*_{11} & f^*_{12} & f^*_{13} & f^*_{14} \\ f^*_{21} & f^*_{22} & f^*_{23} & f^*_{24} \end{bmatrix}$$

where f^*_{11}, f^*_{12} are 1×1 matrices; f^*_{13}, f^*_{14} are 1×2 matrices; f^*_{21}, f^*_{22} are 2×1 matrices; f^*_{23}, f^*_{24} are 2×2 matrices which can be easily written in terms of elements of 3×6 matrix **f**.

From Eq. (4.4), we derive

$$\begin{bmatrix} \Gamma^K_2 f^*_{11} & \Gamma^K_2 f^*_{12} & \Gamma^K_2 f^*_{13} & \Gamma^K_2 f^*_{14} \\ \Gamma^K_5 f^*_{21} & \Gamma^K_5 f^*_{22} & \Gamma^K_5 f^*_{23} & \Gamma^K_5 f^*_{24} \end{bmatrix} = \begin{bmatrix} f^*_{11}\Gamma^K_1 & f^*_{12}\Gamma^K_1 & f^*_{13}\Gamma^K_5 & f^*_{14}\Gamma^K_6 \\ f^*_{21}\Gamma^K_1 & f^*_{22}\Gamma^K_1 & f^*_{23}\Gamma^K_5 & f^*_{24}\Gamma^K_6 \end{bmatrix} \tag{C.2}$$

Comparing the corresponding terms of Eq. (C.2) and applying the Schur's lemma, we find the following results.

(i) There is only one non-zero matrix f^*_{23} and it is of the form $\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
(ii) All other f^*'s vanish.

The system of linear equations derived from (C.2) leads to only one independent constant:

$$f_{14} = -f_{25}.$$

(3) The matrix $\mathbf{Q}^P\mathbf{a}\mathbf{Q}^{P-1}$

Let $\mathbf{a}^* = \mathbf{Q}^P\mathbf{a}\mathbf{Q}^{P-1} = \begin{bmatrix} a^*_{11} & a^*_{12} \\ a^*_{21} & a^*_{22} \end{bmatrix}$

where

$$a^*_{11} = a_{33}; \quad a^*_{12} = [a_{31} \quad a_{32}]; \quad a^*_{21} = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}; \quad a^*_{22} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Eq. (4.4) gives

$$\begin{bmatrix} \Gamma^K_2 a^*_{11} & \Gamma^K_2 a^*_{12} \\ \Gamma^K_5 a^*_{21} & \Gamma^K_5 a^*_{22} \end{bmatrix} = \begin{bmatrix} a^*_{11}\Gamma^K_2 & a^*_{12}\Gamma^K_5 \\ a^*_{21}\Gamma^K_2 & a^*_{22}\Gamma^K_5 \end{bmatrix} \tag{C.3}$$

Comparing the corresponding terms in Eq. (C.3), and applying Schur's lemma, we obtain

(i) a^*_{11} is an independent non-zero constant.
(ii) a^*_{22} is a matrix of the form $\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The 3×3 matrix **a** contains only two non-zero independent constants:

$a_{33}, a_{11} = a_{22}$

(4) The matrix $\mathbf{Q}^P \mathbf{j} \mathbf{Q}^{\Pi-1}$

Let $\mathbf{j}^* = \mathbf{Q}^P \mathbf{j} \mathbf{Q}^{\Pi-1} = \begin{bmatrix} j^*_{11} & j^*_{12} & j^*_{13} & j^*_{14} & j^*_{15} & j^*_{16} \\ j^*_{21} & j^*_{22} & j^*_{23} & j^*_{24} & j^*_{25} & j^*_{26} \end{bmatrix}$

where $j^*_{11}, j^*_{12}, j^*_{13}$ are 1×1 matrices; $j^*_{14}, j^*_{15}, j^*_{16}$ are 1×2 matrices; $j^*_{21}, j^*_{22}, j^*_{23}$ are 2×1 matrices; $j^*_{24}, j^*_{25}, j^*_{26}$ are 2×2 matrices which can be written in terms of elements of 3×9 matrix **j**.

From, Eq. (4.4), we derive

$$\begin{bmatrix} \Gamma^K_2 j^*_{11} & \Gamma^K_2 j^*_{12} & \Gamma^K_2 j^*_{13} & \Gamma^K_2 j^*_{14} & \Gamma^K_2 j^*_{15} & \Gamma^K_2 j^*_{16} \\ \Gamma^K_5 j^*_{21} & \Gamma^K_5 j^*_{22} & \Gamma^K_5 j^*_{23} & \Gamma^K_5 j^*_{24} & \Gamma^K_5 j^*_{25} & \Gamma^K_5 j^*_{26} \end{bmatrix} = \begin{bmatrix} j^*_{11}\Gamma^K_1 & j^*_{12}\Gamma^K_1 & j^*_{13}\Gamma^K_2 & j^*_{14}\Gamma^K_5 & j^*_{15}\Gamma^K_5 & j^*_{16}\Gamma^K_6 \\ j^*_{21}\Gamma^K_1 & j^*_{22}\Gamma^K_1 & j^*_{23}\Gamma^K_2 & j^*_{24}\Gamma^K_5 & j^*_{25}\Gamma^K_5 & j^*_{26}\Gamma^K_6 \end{bmatrix} \tag{C.4}$$

Comparing the corresponding terms in (C.4) and using Schur's lemma, we find

(i) j^*_{13} is a 1×1 non-zero matrix.
(ii) j^*_{24}, j^*_{25} are 2×2 matrices of the form $\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
(iii) All other j^* matrices vanish.

This leads to three independent constants:

$j_{38} = -j_{39}; \quad j_{14} = -j_{27}; \quad j_{15} = -j_{26}$

(5) The matrix $\mathbf{Q}^\Pi \mathbf{d} \mathbf{Q}^{S-1}$

$$\text{Let } \mathrm{d}^* = \mathbf{Q}^\Pi \mathbf{d} \mathbf{Q}^{S-1} = \begin{bmatrix} d^*_{11} & d^*_{12} & d^*_{13} & d^*_{14} \\ d^*_{21} & d^*_{22} & d^*_{23} & d^*_{24} \\ d^*_{31} & d^*_{32} & d^*_{33} & d^*_{34} \\ d^*_{41} & d^*_{42} & d^*_{43} & d^*_{44} \\ d^*_{51} & d^*_{52} & d^*_{53} & d^*_{54} \\ d^*_{61} & d^*_{62} & d^*_{63} & d^*_{64} \end{bmatrix}$$

where d^*_{11}, d^*_{12}, d^*_{21}, d^*_{22}, d^*_{31}, d^*_{32} are 1×1 matrices; d^*_{13}, d^*_{14}, d^*_{23}, d^*_{24}, d^*_{33}, d^*_{34} are 1×2 matrices; d^*_{41}, d^*_{42}, d^*_{51}, d^*_{52}, d^*_{61}, d^*_{62} are 2×1 matrices; d^*_{43}, d^*_{44}, d^*_{53}, d^*_{54}, d^*_{63}, d^*_{64} are 2×2 matrices. These matrices can be written in terms of the elements of 9×6 matrix $\mathbf{d}$.

From Eq. (4.4), we derive

$$\begin{bmatrix} \Gamma_1^K d^*_{11} & \Gamma_1^K d^*_{12} & \Gamma_1^K d^*_{13} & \Gamma_1^K d^*_{14} \\ \Gamma_1^K d^*_{21} & \Gamma_1^K d^*_{22} & \Gamma_1^K d^*_{23} & \Gamma_1^K d^*_{24} \\ \Gamma_2^K d^*_{31} & \Gamma_2^K d^*_{32} & \Gamma_2^K d^*_{33} & \Gamma_2^K d^*_{34} \\ \Gamma_5^K d^*_{41} & \Gamma_5^K d^*_{42} & \Gamma_5^K d^*_{43} & \Gamma_5^K d^*_{44} \\ \Gamma_5^K d^*_{51} & \Gamma_5^K d^*_{52} & \Gamma_5^K d^*_{53} & \Gamma_5^K d^*_{54} \\ \Gamma_6^K d^*_{61} & \Gamma_6^K d^*_{62} & \Gamma_6^K d^*_{63} & \Gamma_6^K d^*_{64} \end{bmatrix} = \begin{bmatrix} d^*_{11}\Gamma_1^K & d^*_{12}\Gamma_1^K & d^*_{13}\Gamma_5^K & d^*_{14}\Gamma_6^K \\ d^*_{21}\Gamma_1^K & d^*_{22}\Gamma_1^K & d^*_{23}\Gamma_5^K & d^*_{24}\Gamma_6^K \\ d^*_{31}\Gamma_1^K & d^*_{32}\Gamma_1^K & d^*_{33}\Gamma_5^K & d^*_{34}\Gamma_6^K \\ d^*_{41}\Gamma_1^K & d^*_{42}\Gamma_1^K & d^*_{43}\Gamma_5^K & d^*_{44}\Gamma_6^K \\ d^*_{51}\Gamma_1^K & d^*_{52}\Gamma_1^K & d^*_{53}\Gamma_5^K & d^*_{54}\Gamma_6^K \\ d^*_{61}\Gamma_1^K & d^*_{62}\Gamma_1^K & d^*_{63}\Gamma_5^K & d^*_{64}\Gamma_6^K \end{bmatrix} \tag{C.5}$$

Comparing the corresponding terms in (C.5) and applying the Schur's lemma, we find that

(i) d^*_{11}, d^*_{12}, d^*_{21}, d^*_{22} are 1×1 non-zero matrices.
(ii) d^*_{43}, d^*_{53}, d^*_{64} are 2×2 matrices of the form $\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
(iii) All other matrices in d^* vanish.

This leads to seven independent non-vanishing constants

$$d_{33};\;\; d_{31} = d_{32};\;\; d_{13} = d_{23};\;\; d_{11} + d_{12};\;\; d_{44} = d_{75};\;\; d_{54} = d_{65};\;\; d_{86} = d_{96} = d_{11} - d_{12}$$

(6) The matrix $\mathbf{Q}^{\Pi}\mathbf{b}\mathbf{Q}^{\Pi-1}$

$$\text{Let } \mathbf{b}^* = \mathbf{Q}^{\Pi}\mathbf{b}\mathbf{Q}^{\Pi-1} = \begin{bmatrix} b^*_{11} & b^*_{12} & b^*_{13} & b^*_{14} & b^*_{15} & b^*_{16} \\ b^*_{21} & b^*_{22} & b^*_{23} & b^*_{24} & b^*_{25} & b^*_{26} \\ b^*_{31} & b^*_{32} & b^*_{33} & b^*_{34} & b^*_{35} & b^*_{36} \\ b^*_{41} & b^*_{42} & b^*_{43} & b^*_{44} & b^*_{45} & b^*_{46} \\ b^*_{51} & b^*_{52} & b^*_{53} & b^*_{54} & b^*_{55} & b^*_{56} \\ b^*_{61} & b^*_{62} & b^*_{63} & b^*_{64} & b^*_{65} & b^*_{66} \end{bmatrix}$$

where

$$b^*_{11}, b^*_{12}, b^*_{13}, b^*_{21}, b^*_{22}, b^*_{23}, b^*_{31}, b^*_{32}, b^*_{33} \text{ are } 1 \times 1 \text{ matrices;}$$

$$b^*_{14}, b^*_{15}, b^*_{16}, b^*_{24}, b^*_{25}, b^*_{26}, b^*_{34}, b^*_{35}, b^*_{36} \text{ are } 1 \times 2 \text{ matrices;}$$

$$b^*_{41}, b^*_{42}, b^*_{43}, b_{51}, b^*_{52}, b^*_{53}, b^*_{61}, b^*_{62}, b^*_{63} \text{ are } 2 \times 1 \text{ matrices;}$$

$$b^*_{44}, b^*_{45}, b^*_{46}, b^*_{54}, b^*_{55}, b^*_{56}, b^*_{64}, b^*_{65}, b^*_{66} \text{ are } 2 \times 2 \text{ matrices :}$$

Eq. (4.4) gives

$$\begin{bmatrix} \Gamma_1^K b_{11}^* & \Gamma_1^K b_{12}^* & \Gamma_1^K b_{13}^* & \Gamma_1^K b_{14}^* & \Gamma_1^K b_{15}^* & \Gamma_1^K b_{16}^* \\ \Gamma_1^K b_{21}^* & \Gamma_1^K b_{22}^* & \Gamma_1^K b_{23}^* & \Gamma_1^K b_{24}^* & \Gamma_1^K b_{25}^* & \Gamma_1^K b_{26}^* \\ \Gamma_2^K b_{31}^* & \Gamma_2^K b_{32}^* & \Gamma_2^K b_{33}^* & \Gamma_2^K b_{34}^* & \Gamma_2^K b_{35}^* & \Gamma_2^K b_{36}^* \\ \Gamma_5^K b_{41}^* & \Gamma_5^K b_{42}^* & \Gamma_5^K b_{43}^* & \Gamma_5^K b_{44}^* & \Gamma_5^K b_{45}^* & \Gamma_5^K b_{46}^* \\ \Gamma_5^K b_{51}^* & \Gamma_5^K b_{52}^* & \Gamma_5^K b_{53}^* & \Gamma_5^K b_{54}^* & \Gamma_5^K b_{55}^* & \Gamma_5^K b_{56}^* \\ \Gamma_6^K b_{61}^* & \Gamma_6^K b_{62}^* & \Gamma_6^K b_{63}^* & \Gamma_6^K b_{64}^* & \Gamma_6^K b_{65}^* & \Gamma_6^K b_{66}^* \end{bmatrix} = \begin{bmatrix} b_{11}^*\Gamma_1^K & b_{12}^*\Gamma_1^K & b_{13}^*\Gamma_2^K & b_{14}^*\Gamma_5^K & b_{15}^*\Gamma_5^K & b_{16}^*\Gamma_6^K \\ b_{21}^*\Gamma_1^K & b_{22}^*\Gamma_1^K & b_{23}^*\Gamma_2^K & b_{24}^*\Gamma_5^K & b_{25}^*\Gamma_5^K & b_{26}^*\Gamma_6^K \\ b_{31}^*\Gamma_1^K & b_{32}^*\Gamma_1^K & b_{33}^*\Gamma_2^K & b_{34}^*\Gamma_5^K & b_{35}^*\Gamma_5^K & b_{36}^*\Gamma_6^K \\ b_{41}^*\Gamma_1^K & b_{42}^*\Gamma_1^K & b_{43}^*\Gamma_2^K & b_{44}^*\Gamma_5^K & b_{45}^*\Gamma_5^K & b_{46}^*\Gamma_6^K \\ b_{51}^*\Gamma_1^K & b_{52}^*\Gamma_1^K & b_{53}^*\Gamma_2^K & b_{54}^*\Gamma_5^K & b_{55}^*\Gamma_5^K & b_{56}^*\Gamma_6^K \\ b_{61}^*\Gamma_1^K & b_{62}^*\Gamma_1^K & b_{63}^*\Gamma_2^K & b_{64}^*\Gamma_5^K & b_{65}^*\Gamma_5^K & b_{66}^*\Gamma_6^K \end{bmatrix} \tag{C.6}$$

Comparing the corresponding terms in Eq. (C.6) and applying the Schur's lemma, we find that

(i) $b_{11}^*, b_{12}^*, b_{21}^*, b_{22}^*, b_{33}^*$ are 1×1 non-zero matrices.
(ii) $b_{44}^*, b_{45}^*, b_{54}^*, b_{55}^*, b_{66}^*$ are 2×2 matrices of the forms $\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
(iii) All other b^* vanish.

This leads to seven independent non-vanishing constants:

$$b_{33}, b_{12} = b_{21},\ \ b_{11} = b_{22},\ \ b_{31} = b_{32},\ \ b_{55} = b_{66},\ \ b_{44} = b_{77},\ \ b_{88} = b_{99} = b_{11} - b_{12};\ \ b_{54} = b_{67}$$

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